Ax-Schanuel and strongly minimal sets in j-reducts of differentially closed fields

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16 May 2018

The j-function

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- $j(gz) = j(z)$ for all $g \in \mathrm{SL}_2(\mathbb{Z})$.
- By means of j the quotient $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ is identified with $\mathbb{C}$ (thus, j is a bijection from the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ to $\mathbb{C}$).

Modular polynomials

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- For $g \in \mathrm{GL}_2^+(\mathbb{Q})$ we let $N(g)$ be the determinant of g scaled so that it has relatively prime integral entries.
- For each positive integer N there is an irreducible polynomial $\Phi_N(X, Y) \in \mathbb{Z}[X, Y]$ such that whenever $g \in \mathrm{GL}_2^+(\mathbb{Q})$ with $N = N(g)$, the function $\Phi_N(j(z), j(gz))$ is identically zero.

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- Conversely, if $\Phi_N(j(x), j(y)) = 0$ for some $x, y \in \mathbb{H}$ then $y = gx$ for some $g \in \mathrm{GL}_2^+(\mathbb{Q})$ with $N = N(g)$.

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- The polynomials Φ_N are called *modular polynomials*.
- $\Phi_1(X, Y) = X - Y$ and all the other modular polynomials are symmetric.
- Two elements $w_1, w_2 \in \mathbb{C}$ are called *modularly independent* if they do not satisfy any modular relation $\Phi_N(w_1, w_2) = 0$.

Differential equation

- The j-function satisfies an order 3 algebraic differential equation over $\mathbb{Q}$. Namely, $F(j, j', j'', j''') = 0$ where

$$F(y_0, y_1, y_2, y_3) = \frac{y_3}{y_1} - \frac{3}{2}\left(\frac{y_2}{y_1}\right)^2 + \frac{y_0^2 - 1968y_0 + 2654208}{2y_0^2(y_0 - 1728)^2} \cdot y_1^2.$$

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- Thus

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where S denotes the *Schwarzian derivative* defined by $Sy = \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2$ and $R(y) = \frac{y^2 - 1968y + 2654208}{2y^2(y-1728)^2}$.

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- All functions $j(gz)$ with $g \in \mathrm{SL}_2(\mathbb{C})$ satisfy the differential equation $F(y, y', y'', y''') = 0$ and in fact all solutions are of that form.

Ax-Schanuel for j

- In a differential field $(K; +, \cdot, ')$ for a non-constant $x \in K$ define a derivation $\partial_x : K \to K$ by $\partial_x : y \mapsto \frac{y'}{x'}$.

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- Let $F(y, y', y'', y''')$ be the differential equation of j. Consider its two-variable version

$$f(x,y) := F(y, \partial_x y, \partial_x^2 y, \partial_x^3 y) = 0.$$

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Theorem (Pila-Tsimerman, 2015)

Let $(z_i, j_i) \in K^2$, $i = 1, \ldots, n$, be non-constant solutions to the above equation. If j_i's are pairwise modularly independent then

$$td_C C(z_1, j_1, \partial_{z_1} j_1, \partial_{z_1}^2 j_1, \ldots, z_n, j_n, \partial_{z_n} j_n, \partial_{z_n}^2 j_n) \geq 3n+1. \quad (1)$$

Ax-Schanuel without derivatives

Corollary

Let $(z_i, j_i) \in K^2, \; i = 1, \ldots, n,$ *be non-constant with* $f(z_i, j_i) = 0$. *If* j_i*'s are pairwise modularly independent then*

$$\operatorname{td}_C C(\bar{z}, \bar{j}) \geq n + 1.$$

Strong minimality of $F(y, y', y'', y''') = 0$

Using the Ax-Schanuel theorem for the j-function one can prove that the set $F(y, y', y'', y''') = 0$ is strongly minimal and geometrically trivial. This is a result of Freitag and Scanlon. I gave a new proof for this result (both proofs are based on Pila's Modular Ax-Lindemann-Weierstrass with Derivatives theorem).

j-reducts of differential fields

If K is a differential field, consider its reduct $K_{E_j} := (K; +, \cdot, E_j)$ where $E_j(x, y)$ is a binary relation interpreted as the set of solutions of the equation $f(x, y) = 0$.

Basic axioms

Definition

The theory T_j^0 consists of the following first-order statements about a structure K in the language $\mathfrak{L}_j := \{+, \cdot, E_j, 0, 1\}$.

- A1 K is an algebraically closed field with an algebraically closed subfield $C := C_K$, which is defined by $E_j(1, y)$. Further, $C^2 \subseteq E_j$.
- A2 If $(z, j) \in E_j$ then for any $g \in \mathrm{SL}_2(C)$ we have $(gz, j) \in E_j$. Conversely, if for some j we have $(z_1, j), (z_2, j) \in E_j$ then $z_2 = gz_1$ for some $g \in \mathrm{SL}_2(C)$.
- A3 If $(z, j_1) \in E_j$ and $\Phi(j_1, j_2) = 0$ for some modular polynomial $\Phi(X, Y)$ then $(z, j_2) \in E_j$.
- AS If $(z_i, j_i) \in E_j$, $i = 1, \ldots, n$, with $\mathrm{td}_C\, C(\bar{z}, \bar{j}) \leq n$ then $\Phi_N(j_i, j_k) = 0$ for some N and $1 \leq i < k \leq n$ or $j_i \in C$ for some i.

Let n be a positive integer, $k \leq n$ and $1 \leq i_1 < \ldots < i_k \leq n$. Denote $\bar{i} := (i_1, \ldots, i_k)$ and define the projection map $\mathrm{pr}_{\bar{i}} : K^{2n} \to K^{2k}$ by

$$\mathrm{pr}_{\bar{i}} : (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (x_{i_1}, \ldots, x_{i_k}, y_{i_1}, \ldots, y_{i_k}).$$

Definition

Let K be an algebraically closed field. An irreducible algebraic variety $V \subseteq K^{2n}$ is *normal* if for any $1 \leq i_1 < \ldots < i_k \leq n$ we have $\dim \mathrm{pr}_{\bar{i}} V \geq k$. We say V is *strongly normal* if the strict inequality $\dim \mathrm{pr}_{\bar{i}} V > k$ holds.

Existential closedness

Consider the following statements.

- EC For each normal variety $V \subseteq K^{2n}$ the intersection $E_j(K) \cap V(K)$ is non-empty.
- NT There is a non-constant element in K.

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Let T_j be the theory A1-A3,AS,EC,NT.

Theorem (A., 2017)

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Conjecture (EC conjecture)

E_j-reducts of differentially closed fields satisfy EC. Hence, T_j is a complete axiomatisation of their first-order theory.

Strongly minimal sets in j-reducts

Let $(K; +, \cdot, ')$ be a countable saturated differentially closed field and $K_{E_j} = (K : +, \cdot, E_j)$ be its j-reduct.

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Theorem (A., 2018)

Assume the EC conjecture. Then all strongly minimal sets in K_{E_j} are either geometrically trivial or non-orthogonal to the field of constants.

Sketch of proof

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- We get a quantifier elimination result: every formula is equivalent to a Boolean combination of existential formulas (near model completeness). Moreover, "small" sets are existentially definable.
- Ax-Schanuel states that any algebraic relation between several solutions of the differential equation of the j-function boils down to a modular relation between TWO solutions (which is binary, hence geometric triviality).

Thank you