Existential Closedness and Zilber-Pink for j, j', j''

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- General model-theoretic context.
- Schanuel is a special case of the generalised period conjecture.

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- Schanuel is a special case of the generalised period conjecture.
- In the differential setting it is natural and often necessary to include derivatives.

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Why are they important?

- General model-theoretic context.
- Schanuel is a special case of the generalised period conjecture.
- In the differential setting it is natural and often necessary to include derivatives.
- Many approaches to these conjectures (even without derivatives) involve techniques where we have to deal with derivatives.

The j-function

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- Let $j : \mathbb{H} \to \mathbb{C}$ be the modular j-function.
- j is holomorphic on $\mathbb{H}$ and is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, i.e. $j(\gamma z) = j(z)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Modular polynomials

- There is a countable collection of irreducible polynomials $\Phi_N \in \mathbb{Z}[X, Y]$ $(N \geq 1)$, called *modular polynomials*, such that for any $z_1, z_2 \in \mathbb{H}$

$$\Phi_N(j(z_1), j(z_2)) = 0 \text{ for some } N \text{ iff } z_2 = gz_1 \text{ for some } g \in \mathrm{GL}_2^+(\mathbb{Q}).$$

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- $\Phi_1(X, Y) = X - Y$ and all the other modular polynomials are symmetric.

The following is a modular analogue of Schanuel's conjecture.

Conjecture (Modular Schanuel Conjecture)

Let $z_1, \ldots, z_n \in \mathbb{H}$ be non-quadratic numbers with distinct $\mathrm{GL}_2^+(\mathbb{Q})$-orbits. Then $\mathrm{td}_{\mathbb{Q}}\, \mathbb{Q}(z_1, \ldots, z_n, j(z_1), \ldots, j(z_n)) \geq n$.

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By abuse of notation we will let j denote all Cartesian powers of itself. Similarly we let $\Gamma_j := \{(\bar{z}, j(\bar{z})) : \bar{z} \in \mathbb{H}^n\} \subseteq \mathbb{C}^{2n}$ be the graph of j in $\mathbb{H}^n \times \mathbb{C}^n$ for any n.

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Conjecture (Modular Existential Closedness)

Let $V \subseteq \mathbb{H}^n \times \mathbb{C}^n$ be an irreducible ***froad*** *(**free** and **broad**) variety defined over $\mathbb{C}$. Then $V \cap \Gamma_j \neq \emptyset$.*

This is an analogue of Zilber's *Exponential Closedness* conjecture.

Froad varieties

We will use the following notation.

- $(n) := (1, \ldots, n)$, and $\bar{k} \subseteq (n)$ means that $\bar{k} = (k_1, \ldots, k_l)$ for some $1 \leq k_1 < \ldots < k_l \leq n$.
- The coordinates of $\mathbb{C}^{2n}$ will be denoted by $(x_1, \ldots, x_n, y_1, \ldots, y_n)$.
- For $\bar{k} = (k_1, \ldots, k_l) \subseteq (n)$ define

$$\mathrm{pr}_{\bar{k}} : \mathbb{C}^n \to \mathbb{C}^l : (x_1, \ldots, x_n) \mapsto (x_{k_1}, \ldots, x_{k_l}),$$

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Definition

Let $V \subseteq \mathbb{C}^{2n}$ be an algebraic variety.

- V is ***broad*** if for any $\bar{k} \subseteq (n)$ of length l we have $\dim \mathrm{pr}_{\bar{k}} V \geq l$.
- V is ***free*** if no coordinate is constant on V, no relation of the form $\Phi_N(y_i, y_k) = 0$ holds on V, and no relation of the form $x_k = gx_i$ holds on V where $g \in \mathrm{GL}_2(\mathbb{Q})$.
- V is ***froad*** if it is free and broad.

Modular EC with Derivatives

Let $J : \mathbb{H} \to \mathbb{C}^3$ be given by $J : z \mapsto (j(z), j'(z), j''(z))$.

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Conjecture (Modular Existential Closedness with Derivatives)

*Let $V \subseteq \mathbb{H}^n \times \mathbb{C}^{3n}$ be an irreducible **froad** variety defined over $\mathbb{C}$. Then $V \cap \Gamma_J \neq \emptyset$.*

Froad varieties

- The coordinates of $\mathbb{C}^{4n}$ will be denoted by $(\bar{x}, \bar{y}, \bar{y}', \bar{y}'')$.
- For a tuple $\bar{k} = (k_1, \ldots, k_l) \subseteq (n)$ define a map

$$\mathrm{pr}_{\bar{k}} : \mathbb{C}^{4n} \to \mathbb{C}^{4l} : (\bar{x}, \bar{y}, \bar{y}', \bar{y}'') \mapsto (\mathrm{pr}_{\bar{k}}(\bar{x}), \mathrm{pr}_{\bar{k}}(\bar{y}), \mathrm{pr}_{\bar{k}}(\bar{y}'), \mathrm{pr}_{\bar{k}}(\bar{y}'')).$$

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Definition

- An algebraic variety $V \subseteq \mathbb{C}^{4n}$ is ***broad*** if for any $\bar{k} \subseteq (n)$ of length l we have $\dim \mathrm{pr}_{\bar{k}} V \geq 3l$.
- $V \subseteq \mathbb{C}^{4n}$ is ***free*** if its projection to the first $2n$ coordinates is free.
- V is ***froad*** if it is free and broad.

MSCD with special points

Definition

- An irreducible subvariety $U \subseteq \mathbb{H}^n$ is called $\mathrm{GL}_2^+(\mathbb{Q})$-***special*** if it is defined by some equations of the form $z_i = g_{i,k} z_k,\ i \neq k$, with $g_{i,k} \in \mathrm{GL}_2^+(\mathbb{Q})$, and some equations of the form $z_i = \tau_i$ where $\tau_i \in \mathbb{H}$ is a quadratic number.
- For a $\mathrm{GL}_2^+(\mathbb{Q})$-special variety U we denote by $\langle U \rangle$ the Zariski closure of the graph of the restriction $J|_U$ (i.e. the set $\{(\bar{z}, J(\bar{z})) : \bar{z} \in U\}$) over $\mathbb{Q}^{\mathrm{alg}}$.
- The $\mathrm{GL}_2^+(\mathbb{Q})$-***special closure*** of an irreducible variety $W \subseteq \mathbb{H}^n$ is the smallest $\mathrm{GL}_2^+(\mathbb{Q})$-special variety containing W.

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Conjecture (MSCD with Special Points)

Let $z_1, \ldots, z_n \in \mathbb{H}$ be arbitrary and let $U \subseteq \mathbb{H}^n$ be the $\mathrm{GL}_2^+(\mathbb{Q})$-special closure of $(z_1, \ldots, z_n)$. Then $\mathrm{td}_{\mathbb{Q}} \mathbb{Q}(z_1, \ldots, z_n, J(z_1), \ldots, J(z_n)) \geq \dim\langle U \rangle - \dim U$.

Defining $\langle U \rangle$ algebraically

The coordinates of $\mathbb{C}^{4n}$ are denoted by $(\bar{x}, \bar{y}, \bar{y}', \bar{y}'')$.
Assume U has no constant coordinates. Let the first two coordinates of U be related, i.e. $x_2 = gx_1$ for some $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$, and let $\Phi(j(z), j(gz)) = 0$ for some modular polynomial Φ.

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$$\frac{\partial \Phi}{\partial Y_1}(j(z), j(gz)) \cdot j'(z) + \frac{\partial \Phi}{\partial Y_2}(j(z), j(gz)) \cdot j'(gz) \cdot \frac{ad - bc}{(cz + d)^2} = 0. \qquad (\star)$$

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Thus, $\langle U \rangle$ satisfies the following equation:

$$\frac{\partial \Phi}{\partial Y_1}(y_1, y_2) \cdot y_1' + \frac{\partial \Phi}{\partial Y_2}(y_1, y_2) \cdot y_2' \cdot \frac{ad - bc}{(cx_1+d)^2} = 0. \qquad (\dagger)$$

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Differentiating once more we will get another equation between $(x_1, x_2, y_1, y_2, y_1', y_2', y_1'', y_2'')$, and we will have four equations defining the projection of $\langle U \rangle$ to the first two coordinates.

Defining $\langle U \rangle$ algebraically

In general, we have a partition of $\{1, \ldots, n\}$ where two indices are in the same partitand if and only if the corresponding coordinates are related on U. We call the projection of $\langle U \rangle$ to a partitand a *block*. Then each block is defined by equations of the form described above and has dimension 4, and $\langle U \rangle$ is the product of its blocks.

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There is a dual Existential Closedness statement for MSCD with special points, but it is equivalent to the other MECD statement.

Modular Zilber-Pink

Definition

- A *j-**special*** variety in $\mathbb{C}^n$ is an irreducible component of a variety defined by some modular equations $\Phi_N(y_k, y_l) = 0$.
- Let $V \subseteq \mathbb{C}^n$ be a variety. A *j-**atypical*** subvariety of V is an atypical component W of an intersection $V \cap T$ where T is j-special. Atypical means $\dim W > \dim V + \dim T - n$.

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Conjecture (Modular Zilber-Pink, MZP)

Let $V \subseteq \mathbb{C}^n$ be an algebraic variety. Let also $\mathrm{Atyp}_j(V)$ be the union of all j-atypical subvarieties of V.

(1) *There is a finite collection Σ of proper j-special subvarieties of $\mathbb{C}^n$ such that every j-atypical subvariety of V is contained in some $T \in \Sigma$.*

(2) *V contains only finitely many maximal j-atypical subvarieties.*

(3) *$\mathrm{Atyp}_j(V)$ is contained in a finite union of proper j-special subvarieties of $\mathbb{C}^n$.*

(4) *$\mathrm{Atyp}_j(V)$ is a Zariski closed subset of V.*

Special varieties for J

Definition

- For a $\mathrm{GL}_2^+(\mathbb{Q})$-special variety U we denote by $\langle\langle U \rangle\rangle$ the Zariski closure of $J(U)$ over $\mathbb{Q}^{\mathrm{alg}}$.
- A J-***special*** subvariety of $\mathbb{C}^{3n}$ is a set of the form $S = \langle\langle U \rangle\rangle$ where U is a $\mathrm{GL}_2^+(\mathbb{Q})$-special subvariety of $\mathbb{H}^n$.

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- J-special varieties are irreducible.
- j-special varieties are bi-algebraic for the j-function, that is, they are the images under j of algebraic varieties (namely, $\mathrm{GL}_2^+(\mathbb{Q})$-special varieties). J-special varieties are not bi-algebraic for J, but they still capture the algebraic properties of the function J.
- The equations defining a J-special variety can be worked out as above since $\langle\langle U\rangle\rangle$ is a projection of $\langle U\rangle$. In particular, a variety $\langle\langle U\rangle\rangle$ is the product of its *blocks* each of which has dimension $0, 1, 3$ or 4. Dimensions 0 and 1 correspond to constant coordinates. A block has dimension 3 if all the $\mathrm{GL}_2^+(\mathbb{Q})$-matrices linking its x-coordinates are upper triangular, and dimension 4 otherwise.

Definition

For a variety $V \subseteq \mathbb{C}^{3n}$ we let the J-***atypical set*** of V, denoted $\mathrm{Atyp}_J(V)$, be the union of all atypical components of intersections $V \cap T$ in $\mathbb{C}^{3n}$ where $T \subseteq \mathbb{C}^{3n}$ is a J-special variety.

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Conjecture (Modular Zilber-Pink with Derivatives, MZPD)

For every algebraic variety $V \subseteq \mathbb{C}^{3n}$ there is a finite collection Σ of proper $\mathrm{GL}_2^+(\mathbb{Q})$-special subvarieties of $\mathbb{H}^n$ such that

$$\mathrm{Atyp}_J(V) \cap J(\mathbb{H}^n) \subseteq \bigcup_{\substack{U \in \Sigma \\ \bar{\gamma} \in \mathrm{SL}_2(\mathbb{Z})^n}} \langle\langle \bar{\gamma} U \rangle\rangle.$$

MZPD for Froad varieties

Definition

For a variety $V \subseteq \mathbb{C}^{3n}$ we let the ***froadly J-atypical set*** of V, denoted $\mathrm{FAtyp}_J(V)$, be the union of all **froad** and atypical components of intersections $V \cap T$ in $\mathbb{C}^{3n}$ where $T \subseteq \mathbb{C}^{3n}$ is a J-special variety.

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Conjecture (Modular Zilber-Pink with Derivatives for Froad varieties, MZPDF)

For every algebraic variety $V \subseteq \mathbb{C}^{3n}$ there is a finite collection Σ of proper $\mathrm{GL}_2^+(\mathbb{Q})$-special subvarieties of $\mathbb{H}^n$ such that

$$\mathrm{FAtyp}_J(V) \subseteq \bigcup_{\substack{U \in \Sigma \\ \bar{\gamma} \in \mathrm{SL}_2(\mathbb{Z})^n}} \langle\langle \bar{\gamma} U \rangle\rangle.$$

Connection between MECD and MZPD

Proposition

(i) *Assume MECDI. Then MZPD implies MZPDF.*
(ii) *Assume MSCDI. Then MZPDF implies MZPD.*

MSCDI and MECDI are MSCD and MECD for the image (rather than the graph) of J.

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Conjecture (MSCDI)

Let $z_1, \ldots, z_n \in \mathbb{H}$ *be arbitrary and let* $U \subseteq \mathbb{H}^n$ *be the* $\mathrm{GL}_2^+(\mathbb{Q})$*-special closure of* $(z_1, \ldots, z_n)$*. Then* $\mathrm{td}_{\mathbb{Q}} \mathbb{Q}(J(z_1), \ldots, J(z_n)) \geq \dim\langle\langle U \rangle\rangle - \dim U$.

Conjecture (MECDI)

Let $V \subseteq \mathbb{C}^{3n}$ *be an irreducible* ***froad*** *variety. Then* $V \cap \mathrm{Im}(J) \neq \emptyset$.

Let $(F; +, \cdot, D_1, \ldots, D_m)$ be a differential field with an algebraically closed constant field $C = \bigcap_{k=1}^{m} \ker D_k$.

Theorem (Ax-Schanuel with Derivatives)

Let $(F; +, \cdot, D_1, \ldots, D_m)$ be a differential field with commuting derivations and with field of constants C. Let also $(z_i, j_i, j_i', j_i'') \in \mathrm{D}\,\Gamma_J^{\times}(F)$, $i = 1, \ldots, n$. If the j_i's are pairwise modularly independent then $\mathrm{td}_C\, C\left(\bar{z}, \bar{j}, \bar{j}', \bar{j}''\right) \geq 3n + \mathrm{rk}(D_k z_i)_{i,k}$.

Differential/functional versions of MSCD and MECD

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Theorem (Differential MECD)

Let F be a differential field, and $V \subseteq F^{4n}$ be a J-broad variety. Then there is a differential field extension $K \supseteq F$ such that $V(K) \cap \mathrm{D}\,\Gamma_J(K) \neq \emptyset$. In particular, if F is differentially closed then $V(F) \cap \mathrm{D}\,\Gamma_J(F) \neq \emptyset$.

Differential/functional versions of MZPD

Theorem (Functional MZPD; FMZPD)

Let $(K;+,\cdot,D)$ be a differential field with an algebraically closed field of constants C. Given an algebraic variety $V \subseteq C^{3n}$, there is a finite collection Σ of proper j-special subvarieties of C^n such that

$$\mathrm{Atyp}_{\mathrm{D}_J}(V)(K) \cap \mathrm{D}\,\mathrm{Im}_J^{\times}(K) \subseteq \bigcup_{S \sim \Sigma} S.$$

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Theorem (Functional MZPDF; FMZPDF)

Let C be an algebraically closed field of characteristic zero. Given an algebraic variety $V \subseteq C^{3n}$, there is a finite collection Σ of proper j-special subvarieties of C^n such that

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These two are equivalent due to Differential MECDI. In fact, the proof also uses Differential MECDI.