The Existential Closedness with Derivatives conjecture for the j-function

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Schanuel's conjecture and Exponential Closedness

Conjecture (Schanuel, 1960s)

For any $\mathbb{Q}$-linearly independent complex numbers $z_1, \ldots, z_n$ we have

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Conjecture (Exponential Closedness, Zilber, early 2000s)

*Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be a **free** and **rotund** variety. Then V contains a point of the form $(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n})$.*

The j-function

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- j is holomorphic on $\mathbb{H}$ and is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, i.e. $j(\gamma z) = j(z)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Modular polynomials

- There is a countable collection of irreducible polynomials $\Phi_N \in \mathbb{Z}[X, Y]$ ($N \geq 1$), called *modular polynomials*, such that for any $z_1, z_2 \in \mathbb{H}$

$$\Phi_N(j(z_1), j(z_2)) = 0 \text{ for some } N \text{ iff } z_2 = gz_1 \text{ for some } g \in \mathrm{GL}_2^+(\mathbb{Q}).$$

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- $\Phi_1(X, Y) = X - Y$ and all the other modular polynomials are symmetric.

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Conjecture (Modular Existential Closedness, A.-Kirby 2021)

Let $V \subseteq \mathbb{H}^n \times \mathbb{C}^n$ be an irreducible ***froad*** *(****free*** *and* ***broad****) variety defined over $\mathbb{C}$. Then $V \cap \Gamma_j \neq \emptyset$.*

This is an analogue of Zilber's *Exponential Closedness* conjecture.

Froad varieties

The coordinates of $\mathbb{C}^{2n}$ (and $\mathbb{H}^n \times \mathbb{C}^n$) are denoted by $(x_1, \ldots, x_n, y_1, \ldots, y_n)$.

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Definition

Let $V \subseteq \mathbb{H}^n \times \mathbb{C}^n$ be an algebraic variety.

- V is ***broad*** if for any $1 \leq k_1 < \ldots < k_m \leq n$ the projection of V to the coordinates $(x_{k_1}, \ldots, x_{k_m}, y_{k_1}, \ldots, y_{k_m})$ is at least m.

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- V is ***free*** if no coordinate is constant on V, no relation of the form $\Phi_N(y_i, y_k) = 0$ holds on V, and no relation of the form $x_k = gx_i$ holds on V where $g \in \mathrm{GL}_2(\mathbb{Q})$.

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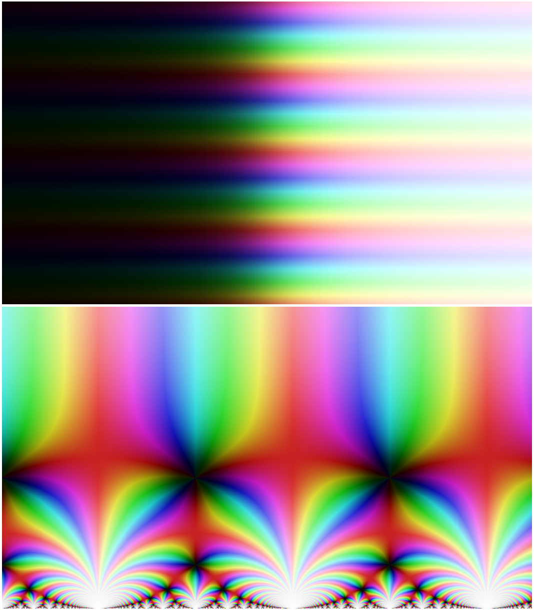
- V is ***froad*** if it is free and broad.

Exponential vs modular EC

There are two main differences between exponential and modular Existential Closedness.

- Exponential functions are defined on the whole complex plane while modular functions are defined on the upper half-plane. These spaces ($\mathbb{C}$ and $\mathbb{H}$) are geometrically different which accounts for different approaches to EC in these two settings.

Exp vs j



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- Modular functions satisfy third-order differential equations, so we can consider EC for these functions together with their first two derivatives. Exponential functions satisfy first-order differential equations so considering derivatives would not change anything.

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- $V \subseteq \mathbb{C}^{4n}$ is ***free*** if its projection to the first $2n$ coordinates is free.
- V is ***froad*** if it is free and broad.

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- Derivatives of modular functions are modular forms of weight 2.

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Some ingredients in the proofs: complex analysis/geometry (Rouché, argument principle, open mapping), o-minimality (dimension theory), differential algebra (differential forms, Ax-Schanuel).