A remark on unlikely intersections

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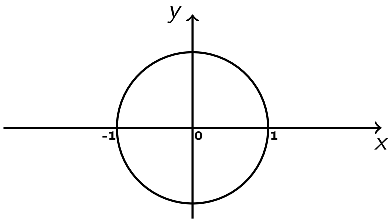

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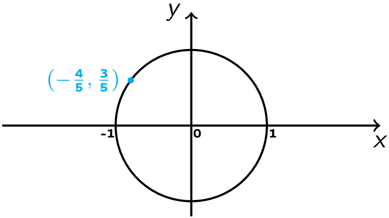

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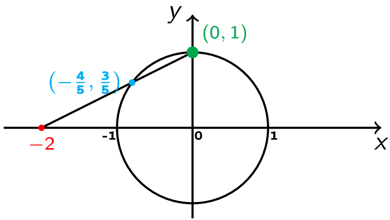

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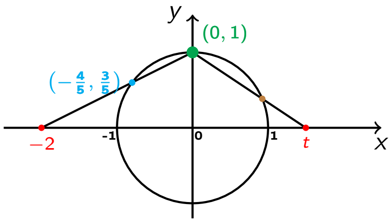

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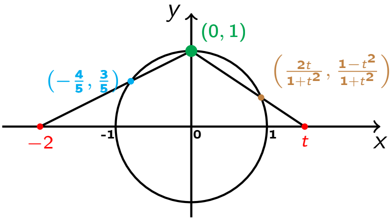

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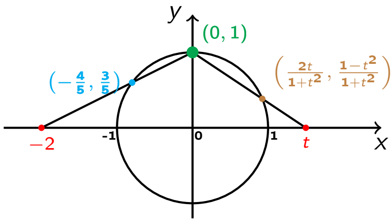

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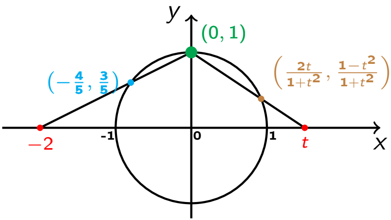

- A famous example is Faltings's theorem (the Mordell conjecture) stating that certain Diophantine equations have only finitely many rational solutions. For instance, the equation $x^4 + y^4 = 1$ has only finitely many rational solutions.

Diophantine geometry (continued)

- We are often interested in special solutions of polynomial equations.
- Examples of special points are roots of unity, i.e. numbers $\xi \in \mathbb{C}$ for which $\xi^n = 1$ for some $n > 0$ (e.g. $i^4 = 1$). These are the images of rational numbers under the function $e^{2\pi iz}$. Indeed, $\left(e^{2\pi i \cdot \frac{m}{n}}\right)^n = (e^{2\pi i})^m = 1$.

Example

- The equation $x^2 + y = -2$ has only finitely many solutions with x, y roots of unity. In fact, the only solutions are $x = \pm i, y = -1$.
- But $x^2 y = 1$ has infinitely many special solutions. If $x = \zeta$ is any root of unity then so is $y = \zeta^{-2}$.

Theorem (Ihara, Serre, Tate)

Let f be an irreducible polynomial. Assume $f(x, y) = 0$ contains infinitely many points (ξ, η) whose coordinates are roots of unity. Then up to multiplication by a constant f is of the form $x^m y^n - \zeta$ where $m, n \in \mathbb{Z}$ and ζ is a root of unity.
In other words, if a curve contains infinitely many points with special coordinates, then it must be of a special form.

Algebraic varieties

- An algebraic variety is a subset of $\mathbb{C}^n$ defined by several polynomial equations. Throughout the talk we may assume $n \leq 3$ and will let x, y, z denote the coordinates on $\mathbb{C}^3$.
- For example, the set

$$\{(x, y, z) \in \mathbb{C}^3 : x^3z^2 + y^3 - z^3 - 1 = 0,\ x^2 + y^2 + xz^4 = 0\}$$

  is an algebraic variety.
- An algebraic variety V is irreducible if it cannot be decomposed into a union of two proper algebraic subvarieties. For instance, in $\mathbb{C}^2$ the variety $x^2 + y^2 = 1$ is irreducible but $x^2 + y^2 = 0$ is reducible (it is the union of the lines $x = iy$ and $x = -iy$).
- If $f(X, Y, Z) \in \mathbb{C}[X, Y, Z]$ is an irreducible polynomial then $f(x, y, z) = 0$ defines an irreducible (hyper)surface.
- Every algebraic variety can be decomposed into a finite union of irreducible components.
- The set $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ can be identified with the variety

$$\{(x, y) \in \mathbb{C}^2 : xy = 1\} \subseteq \mathbb{C}^2.$$

Dimension

- The dimension of V, written $\dim V$, is the maximal length d of chains $V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_d \subseteq V$ of irreducible subvarieties.
- For instance, a point has dimension 0, for it has no proper non-empty subsets. A curve has dimension one as the only proper irreducible subvarieties are points. A surface has dimension 2 since we can choose a curve on it and a point on the curve to get a chain with $d = 2$.
- $\dim \mathbb{C}^n = \dim(\mathbb{C}^\times)^n = n$.
- $\dim V = 0$ if and only if V is finite.
- If $V \subseteq \mathbb{C}^n$ is defined by t independent equations, then we expect its dimension to be $n - t$. For instance, if V is defined by a single non-constant polynomial (it is a hypersurface), then it has dimension $n - 1$. The equation $x = y$ defines a 1-dimensional variety in $\mathbb{C}^2$ and a 2-dimensional variety in $\mathbb{C}^3$.
- The variety defined by three equations $x^2 - y^2 = 1,\ x^2 - z^2 = 1,\ x(y - z) = 0$ has dimension 1 in $\mathbb{C}^3$.

Algebraic tori

- Let $(\mathbb{C}^{\times}; \cdot, 1)$ be the multiplicative group of non-zero complex numbers. It is an algebraic group, i.e. an algebraic variety where the group operation is given by a polynomial map (in this case $(x, y) \mapsto xy$).
- Special points (roots of unity) are the torsion elements of this group, i.e. elements of finite order.
- For any $n \in \mathbb{N}$ the Cartesian power $(\mathbb{C}^{\times})^n$ is also a group under coordinate-wise multiplication. Special points in $(\mathbb{C}^{\times})^n$ are the torsion elements of this group, i.e. tuples of roots of unity.
- A subvariety $T \subseteq (\mathbb{C}^{\times})^n$ is called an algebraic torus if it is irreducible and is a subgroup of $(\mathbb{C}^{\times})^n$.
- For example, the variety $x^5yz^2 = 1$ is an algebraic torus, for if $x_1^5y_1z_1^2 = 1$ and $x_2^5y_2z_2^2 = 1$ then $(x_1x_2)^5 \cdot (y_1y_2) \cdot (z_1z_2)^2 = 1$.
- An algebraic torus is defined by (several) multiplicative equations as above.
- Torsion cosets of tori, that is, sets of the form $\zeta \cdot T$ where $T \subseteq (\mathbb{C}^{\times})^n$ is a torus and ζ is special, are known as special varieties. For example, $x^5yz^3 = i$ is special.
- Special varieties contain infinitely many special points. If an irreducible curve contains infinitely many special points, then it must be special.

Manin-Mumford conjecture

Theorem (Manin-Mumford for tori; Raynaud, Hindry)

Let $V \subseteq (\mathbb{C}^\times)^n$ be an algebraic variety. Then V contains only finitely many maximal special subvarieties.

- If V is an irreducible curve then either it is special or it contains only finitely many special points.
- If V is irreducible and contains a "Zariski dense" set of special points (too many special points) then V is special.

Example

Let $V \subseteq (\mathbb{C}^\times)^3$ be defined by $x^3y^6 + y^2z^3 = 2$. Then the following are the maximal special subvarieties of V:

$$S_1 : xy^2 = 1,\ y^2z^3 = 1,$$
$$S_2 : xy^2 = e^{2\pi i/3},\ y^2z^3 = 1,$$
$$S_3 : xy^2 = e^{4\pi i/3},\ y^2z^3 = 1.$$

Dimension of intersection

- Given two varieties V and W in $\mathbb{C}^n$, one expects

$$\dim(V \cap W) = \dim V + \dim W - n.$$

- For instance, in $\mathbb{C}^3$ two planes (linear subspaces of dim 2) intersect in a line (dim 1) unless the two planes are the same.
- Suppose V is defined by t equations and W is defined by s equations. Then $V \cap W$ is defined by $t + s$ equations, so we expect

$$\dim V = n - t, \ \dim W = n - s, \ \dim(V \cap W) = n - (s + t) = (n - t) + (n - s) - n.$$

- When $\dim V + \dim W < n$, V and W are not expected to intersect. Two curves in a two-dimensional space are likely to intersect, while two curves in a three-dimensional space are not. If they do intersect, then we have an unlikely intersection.

Definition

X is an atypical component of $V \cap W$ if $\dim X > \dim V + \dim W - n$. Note that we always have $\dim X \geq \dim V + \dim W - n$.

Special and atypical subvarieties

Definition

Torsion cosets of tori are special varieties. In $(\mathbb{C}^\times)^3$ these are defined by equations of the form $x^a y^b z^c = \zeta$ where ζ is a root of unity and $a, b, c \in \mathbb{Z}$.

Definition

For a variety $V \subseteq (\mathbb{C}^\times)^n$ and a special variety $S \subseteq (\mathbb{C}^\times)^n$, an irreducible component X of the intersection $V \cap S$ is an atypical subvariety of V if

$$\dim X > \dim V + \dim S - n.$$

Example

- If $V \subseteq (\mathbb{C}^\times)^3$ is defined by the equations $xy + x^2z^3 = i + 1,\ x^5 + xy^6 + y^2 + yz^3 = i - 1$ then its intersection with the special variety $xy = i,\ x^2z^3 = 1$ is atypical. Indeed, the intersection is non-empty (it contains the point $(1, i, 1)$), and in fact has dimension 0.
- If $T \subseteq V \subsetneq (\mathbb{C}^\times)^n$ and T is special then it is an atypical subvariety of V, for $\dim T > \dim V + \dim T - n$.

Conjecture on Intersections with Tori

Conjecture (CIT; Zilber, Bombieri-Masser-Zannier, Pink)

Every algebraic variety in $(\mathbb{C}^\times)^n$ contains only finitely many maximal atypical subvarieties.

Remark

Since special subvarieties of V are atypical, CIT implies Manin-Mumford.

The following is a special case of CIT.

Theorem (Bombieri-Masser-Zannier, Maurin)

Let $V \subseteq (\mathbb{C}^\times)^3$ be a curve not contained in a proper special subvariety of $(\mathbb{C}^\times)^3$. Then V contains only finitely many points (a_1, a_2, a_3) which satisfy two independent multiplicative relations.

Weakly special varieties and closures

Definition

- Arbitrary cosets of algebraic tori are called weakly special varieties. For instance, in $(\mathbb{C}^\times)^3$ the variety $xyz^6 = \pi$ is weakly special.
- Let $X \subseteq (\mathbb{C}^\times)^n$. The (weakly) special closure of X is the smallest (weakly) special subvariety containing X.

Example

- Let $V \subseteq (\mathbb{C}^\times)^3$ be defined by $x + y = 1,\ xy^3z^2 = \pi$. Then the weakly special closure of V is the coset $xy^3z^2 = \pi$. The special closure of V is $(\mathbb{C}^\times)^3$.
- Let $V \subseteq (\mathbb{C}^\times)^3$ be defined by $x + y = 1,\ xy^3z^2 = i$. Then the weakly special closure of V is equal to its special closure and is defined by $xy^3z^2 = i$.
- Let $V \subseteq (\mathbb{C}^\times)^3$ be the point (π, π^2, π^3) (defined by $x = \pi, y = \pi^2, z = \pi^3$). Then the weakly special closure of V is V itself, while its special closure is the torus $y = x^2, z = x^3$.

A weak version of CIT

Theorem (A.)

Every variety $V \subseteq (\mathbb{C}^\times)^n$ contains only finitely many maximal atypical subvarieties whose weakly special closures are special.
In particular, V contains only finitely many maximal atypical subvarieties which contain a special point.

Remark

In $(\mathbb{C}^\times)^3$ this is just Manin-Mumford and does not imply the theorem of Bombieri-Masser-Zannier and Maurin. For $n > 3$ this is stronger than Manin-Mumford.

Proof sketch

Theorem (A.)

For every variety $V \subseteq (\mathbb{C}^{\times})^n$ there is a finite collection Σ of proper special subvarieties of $(\mathbb{C}^{\times})^n$ such that every atypical subvariety of V, whose weakly special closures is special, is contained in some $T \in \Sigma$.

- By Weak/Functional/Geometric CIT, an atypical subvariety X of V is contained in a coset of a torus T from a finite collection of tori.
- The set
$$C := \{c \in (\mathbb{C}^{\times})^n : V \cap cT \text{ is atypical in } (\mathbb{C}^{\times})^n\}$$
is a proper Zariski closed subset of $(\mathbb{C}^{\times})^n$. Roughly, this is because generic varieties intersect typically.
- Now if the weakly special closure of X is special, then X is contained in a torsion coset of T. So we are looking for torsion points in C.
- By Manin-Mumford, all torsion points in C are contained in finitely many maximal special subvarieties of C. Now combine these with the finite collection of tori given by weak CIT.

Generalisations and analogues

- There is a generalisation of the Manin-Mumford conjecture, known as Mordell-Lang (Faltings, Vojta, McQuillan,...). It deals with general semi-abelian varieties instead of algebraic tori and arbitrary finite ranks subgroups instead of torsion subgroups. It can be combined with Weak CIT to produce a stronger theorem.
- There is an analogue of Manin-Mumford in the modular setting, known as André-Oort (Pila). Similarly, there is a modular Mordell-Lang (Habegger-Pila). I proved analogous results in this setting.
- With Chris Daw (Reading) we generalised the above to Shimura varieties where André-Oort (Pila-Shankar-Tsimerman) and Mordell-Lang (in the form of André-Pink-Zannier, Richard-Yafaev) are now known.
- The analogue of CIT in these setting is known as Zilber-Pink. Its weak/functional/geometric version is a consequence of the Ax-Schanuel theorem in the appropriate setting.

Thank you