Functional Modular Zilber-Pink with Derivatives

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- $j(gz) = j(z)$ for all $g \in \mathrm{SL}_2(\mathbb{Z})$.
- By means of j the quotient $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ is identified with $\mathbb{C}$ (thus, j is a bijection from the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ to $\mathbb{C}$).

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- Conversely, if $\Phi_N(j(x), j(y)) = 0$ for some $x, y \in \mathbb{H}$ then $y = gx$ for some $g \in \mathrm{GL}_2^+(\mathbb{Q})$ with $N = N(g)$.

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- The polynomials Φ_N are called *modular polynomials*.
- $\Phi_1(X, Y) = X - Y$ and all the other modular polynomials are symmetric.
- Two elements $w_1, w_2 \in \mathbb{C}$ are called *modularly independent* if they do not satisfy any modular relation $\Phi_N(w_1, w_2) = 0$.

j-special varieties

Definition

A *j-special* subvariety of $\mathbb{C}^n$ (coordinatised by $\bar{y}$) is an irreducible component of a variety defined by modular equations, i.e. equations of the form $\Phi_N(y_i, y_k) = 0$ for some $1 \leq i, k \leq n$ where $\Phi_N(X, Y)$ is a modular polynomial.

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Definition

A subvariety $U \subseteq \mathbb{H}^n$ (i.e. an intersection of $\mathbb{H}^n$ with some algebraic variety) is called *$\mathbb{H}$-special* if it is defined by some equations of the form $z_i = g_{i,k} z_k, \ i \neq k$, with $g_{i,k} \in \mathrm{GL}_2^+(\mathbb{Q})$, and some equations of the form $z_i = \tau_i$ where $\tau_i \in \mathbb{H}$ is a quadratic number. For such a U the image $j(U)$ is j-special (j is identified with its Cartesian powers).

Definition

For a variety $V \subseteq \mathbb{C}^n$ and a special variety $S \subseteq \mathbb{C}^n$, a component X of the intersection $V \cap S$ is an *atypical* subvariety of V if

$$\dim X > \dim V + \dim S - n.$$

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Conjecture (Modular Zilber–Pink)

Every algebraic variety in $\mathbb{C}^n$ contains only finitely many maximal atypical subvarieties.

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Theorem (Pila-Tsimerman, 2015)

Every algebraic variety in $\mathbb{C}^n$ contains only finitely many maximal strongly atypical subvarieties.

J-special varieties

Define a function $J : \mathbb{H} \to \mathbb{C}^3$ by

$$J : z \mapsto (j(z), j'(z), j''(z)).$$

We extend J to $\mathbb{H}^n$ by defining

$$J : \bar{z} \mapsto (j(\bar{z}), j'(\bar{z}), j''(\bar{z}))$$

where $j^{(k)}(\bar{z}) = (j^{(k)}(z_1), \ldots, j^{(k)}(z_n))$ for $k = 0, 1, 2.$ Note that $j'''(z)$ is algebraic over j, j', j''.

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Definition (Pila)

Let $U \subseteq \mathbb{H}^n$ be $\mathbb{H}$-*special*. We denote by $\langle\langle U \rangle\rangle \subseteq \mathbb{C}^{3n}$ the Zariski closure of $J(U)$ over $\mathbb{Q}^{\mathrm{alg}}$. These are the *$J$-special* varieties in $\mathbb{C}^{3n}$.

Remark

J-special varieties are irreducible. Strongly J-special varieties (no constant coordinates) are equal to the product of j-blocks (where all j-coordinates are pairwise modularly related) each of which has dimension 3 or 4.

Modular Zilber-Pink with Derivatives

Definition

For a variety $V \subseteq \mathbb{C}^{3n}$ we let the *J-atypical set* of V, denoted $\operatorname{Atyp}_J(V)$, be the union of all atypical components of intersections $V \cap T$ in $\mathbb{C}^{3n}$ where $T \subseteq \mathbb{C}^{3n}$ is a J-special variety.

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Conjecture (Pila, "MZPD")

For every algebraic variety $V \subseteq \mathbb{C}^{3n}$ there is a finite collection Σ of proper $\mathbb{H}$-special subvarieties of $\mathbb{H}^n$ such that

$$\mathrm{Atyp}_J(V) \cap J(\mathbb{H}^n) \subseteq \bigcup_{\substack{U \in \Sigma \\ \bar{\gamma} \in \mathrm{SL}_2(\mathbb{Z})^n}} \langle\langle \bar{\gamma} U \rangle\rangle.$$

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Remark

Here we may need infinitely many J-special subvarieties to cover the atypical set of V.

Weak Modular Zilber-Pink with Derivatives

Definition

For a J-special variety $T \subseteq \mathbb{C}^{3n}$ and an algebraic variety $V \subseteq \mathbb{C}^{3n}$ an atypical component X of an intersection $V \cap T$ in $\mathbb{C}^{3n}$ is a *strongly J-atypical* subvariety of V if for every irreducible analytic component Y of $X \cap J(\mathbb{H}^n)$, no coordinate is constant on Y. The *strongly J-atypical set* of V, denoted $\mathrm{SAtyp}_J(V)$, is the union of all strongly J-atypical subvarieties of V.

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Theorem (A., 2019)

For every algebraic variety $V \subseteq \mathbb{C}^{3n}$ there is a finite collection Σ of proper $\mathbb{H}$-special subvarieties of $\mathbb{H}^n$ such that

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Sketch of Proof - Complex Ax-Schanuel

- Let $\text{pr}_j : \mathbb{C}^{3n} \to \mathbb{C}^n$ be the projection onto the j-coordinates, i.e. the first n coordinates. By abuse of notation, we also let $\text{pr}_j : \mathbb{C}^{4n} \to \mathbb{C}^n$ be the projection onto the second n coordinates.

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- Let $\Gamma \subseteq \mathbb{H}^n \times \mathbb{C}^{3n}$ be the graph of $J : \mathbb{H}^n \to \mathbb{C}^{3n}$.

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Theorem (Complex Ax-Schanuel for j, Pila-Tsimerman 2015)

Let $V \subseteq \mathbb{C}^{4n}$ be an algebraic variety and let A be an analytic component of the intersection $V \cap \Gamma$. If $\dim A > \dim V - 3n$ and no coordinate is constant on $\text{pr}_j A$ then it is contained in a proper j-special subvariety of $\mathbb{C}^n$.

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Theorem (Uniform Ax-Schanuel)

Let $V_{\bar{c}} \subseteq \mathbb{C}^{4n}$ be a parametric family of algebraic varieties. Then there is a finite collection Σ of proper j-special subvarieties of $\mathbb{C}^n$ such that for every $\bar{c} \subseteq \mathbb{C}$, if $A_{\bar{c}}$ is an analytic component of the intersection $V_{\bar{c}} \cap \Gamma$ with $\dim A_{\bar{c}} > \dim V_{\bar{c}} - 3n$, and no coordinate is constant on $\mathrm{pr}_j A_{\bar{c}}$, then $\mathrm{pr}_j A_{\bar{c}}$ is contained in some $T' \in \Sigma$.

Sketch of Proof - Dimension of Intersection

Theorem (Dimension of Intersection)

Let $A, B \subseteq M$ be analytic varieties where M is smooth. Then for any component X of $A \cap B$ we have

$$\dim X \geq \dim A + \dim B - \dim M.$$

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Corollary

Let $A, B \subseteq M$ be irreducible analytic varieties (M may not be smooth). If X (a component of $A \cap B$) contains a non-singular point of M then

$$\dim X \geq \dim A + \dim B - \dim M.$$

Sketch of proof - Non-singular points

Lemma

Assume $T \subseteq \mathbb{C}^{3n}$ is J-special and Y is a complex analytically irreducible subset of $T \cap J(\mathbb{H}^n)$ without constant coordinates. Then Y contains a non-singular point of T.

Sketch of proof - Non-singular points

Lemma

Assume $T \subseteq \mathbb{C}^{3n}$ is J-special and Y is a complex analytically irreducible subset of $T \cap J(\mathbb{H}^n)$ without constant coordinates. Then Y contains a non-singular point of T.

Proof.

- Assume T consists of a single j-block, i.e. all j-coordinates of T are pairwise modularly related.
- Let $T_s \subsetneq T$ be the set of singular points of T.
- If $Z := J^{-1}(T_s)$ is uncountable then it has a limit point, and we can deduce that $z, j(z), j'(z), j''(z)$ are algebraically dependent. Contradiction.
- Hence $T_s \cap J(\mathbb{H}^n)$ is countable and $Y \not\subseteq T_s$.

□

Theorem recalled

Definition

For a J-special variety $T \subseteq \mathbb{C}^{3n}$ and an algebraic variety $V \subseteq \mathbb{C}^{3n}$ an atypical component X of an intersection $V \cap T$ in $\mathbb{C}^{3n}$ is a *strongly J-atypical* subvariety of V if for every irreducible analytic component Y of $X \cap J(\mathbb{H}^n)$, no coordinate is constant on Y. The *strongly J-atypical set* of V, denoted $\mathrm{SAtyp}_J(V)$, is the union of all strongly J-atypical subvarieties of V.

Theorem

For every algebraic variety $V \subseteq \mathbb{C}^{3n}$ there is a finite collection Σ of proper $\mathbb{H}$-special subvarieties of $\mathbb{H}^n$ such that

$$\mathrm{SAtyp}_J(V) \cap J(\mathbb{H}^n) \subseteq \bigcup_{\substack{U \in \Sigma \\ \bar{\gamma} \in \mathrm{SL}_2(\mathbb{Z})^n}} \langle\langle \bar{\gamma} U \rangle\rangle.$$

Sketch of Proof

- Let $T = \langle\langle U \rangle\rangle \subseteq \mathbb{C}^{3n}$ be a J-special variety and $X \subseteq V \cap T$ be a strongly atypical component, $\dim X > \dim V + \dim T - 3n$.
- Assume $A \subseteq X \cap J(\mathbb{H}^n)$ is an analytic component such that no coordinate is constant on A. Then $A \subseteq J(U) \subseteq T$, and A is an analytic component of $X \cap J(U)$.
- By Lemma, A contains a non-singular point of T. Hence,

$$\dim A \geq \dim X + \dim J(U) - \dim T >$$
$$\dim V + \dim T - 3n + \dim J(U) - \dim T = \dim V + \dim U - 3n.$$

- This implies

$$\dim((U \times A) \cap \Gamma) = \dim A > \dim(U \times V) - 3n.$$

Now the desired result follows from Uniform Ax-Schanuel applied to the parametric family of algebraic varieties $W_{\bar{c}} \times V$ where $W_{\bar{c}}$ varies over the parametric family of all $\mathbb{C}$-geodesic varieties.

Differential equation

- The j-function satisfies an order 3 algebraic differential equation over $\mathbb{Q}$. Namely, $\Psi_j(j, j', j'', j''') = 0$ where

$$\Psi_j(y_0, y_1, y_2, y_3) = \frac{y_3}{y_1} - \frac{3}{2}\left(\frac{y_2}{y_1}\right)^2 + \frac{y_0^2 - 1968y_0 + 2654208}{2y_0^2(y_0 - 1728)^2} \cdot y_1^2.$$

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- Thus

$$\Psi_j(y, y', y'', y''') = Sy + R(y)(y')^2,$$

where S denotes the *Schwarzian derivative* defined by
$Sy = \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2$ and $R(y) = \frac{y^2 - 1968y + 2654208}{2y^2(y-1728)^2}$.

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- All functions $j(gz)$ with $g \in \mathrm{SL}_2(\mathbb{C})$ satisfy the differential equation $\Psi_j(y, y', y'', y''') = 0$ and in fact all solutions are of that form.

Differential equation

Let $(K;+,\cdot,D)$ be a differential field with field of constants $C := \ker D$.

- Let $E_{(z,J)}(x,y,y',y'')$ denote the formula

$$\exists y''' \left(\Psi_j \left(y,y',y'',y''' \right) = 0 \wedge Dx = \frac{Dy}{y'} = \frac{Dy'}{y''} = \frac{Dy''}{y'''} \right).$$

  By abuse of notation we will also let $E_{(z,J)}(K)$ denote the set of all tuples $(\bar{x},\bar{y},\bar{y}',\bar{y}'') \in K^{4n}$ with $(x_i,y_i,y_i',y_i'') \in E_{(z,J)}(K)$. The set $E^{\times}_{(z,J)}(K)$ consists of all $E_{(z,J)}(K)$-points that do not have any constant coordinates.

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- $E_J(y,y',y'')$ is the projection of $E_{(z,J)}$ onto the last three coordinates, i.e. $\exists x E_{(z,J)}(x,y,y',y'')$. Equivalently, E_J is given by

$$\exists y''' \left(\Psi_j(y,y',y'',y''') = 0 \wedge \frac{Dy}{y'} = \frac{Dy'}{y''} = \frac{Dy''}{y'''} \right).$$

  As above, $E_J(K)$ also denotes the set of all tuples $(\bar{y},\bar{y}',\bar{y}'') \in K^{3n}$ such that $(y_i,y_i',y_i'') \in E_J(K)$ for all i, and $E_J^{\times}(K)$ is the set of all points in $E_J(K)$ with no constant coordinates.

Functional equations

Let $E_{(z,j)}(x,y)$ be the projection $\exists y', y'' E_{(z,J)}(x,y,y',y'')$. Define $E^{\times}_{(z,j)}$ as above.

- If $(z_i, j_i) \in E^{\times}_{(z,j)}(K),\ \ i = 1, 2$, and $\Phi_N(j_1, j_2) = 0$ for some modular polynomial Φ_N then $z_2 = gz_1$ for some $g \in \mathrm{SL}_2(C)$.
- If $(z_1, j_1) \in E^{\times}_{(z,j)}(K)$ and $(z_2, j_2) \in K^2$ such that $\Phi_N(j_1, j_2) = 0$ for some Φ_N and $z_2 = gz_1$ for some $g \in \mathrm{SL}_2(C)$ then $(z_2, j_2) \in E^{\times}_{(z,j)}(K)$.

Theorem (Pila-Tsimerman, 2015)

Let $(K; D)$ *be a differential field with field of constants* C*. Assume* $(z_i, j_i, j_i', j_i'') \in E^{\times}_{(z,J)}(K),\ i = 1, \ldots, n$*. If the* j_i*'s are pairwise modularly independent then*

$$\operatorname{td}_C C\left(\bar{z}, \bar{j}, \bar{j}', \bar{j}''\right) \geq 3n + 1.$$

D-special varieties

Let C be an algebraically closed field. Define D as the zero derivation on C and extend $(C; +, \cdot, D)$ to a differentially closed field $(K; +, \cdot, D)$.

- A *C-geodesic variety* $U \subseteq C^n$ (with coordinates $\bar{x}$) is an irreducible component of a variety defined by equations of the form $x_i = g_{i,k} x_k$ for some $g_{i,k} \in \mathrm{SL}_2(C)$. If $S \subseteq C^n$ (with coordinates $\bar{y}$) is a j-special variety, then U is said to be a C-geodesic variety associated with S if for any $1 \leq i, k \leq n$ we have $\Phi_N(y_i, y_k) = 0$ on S for some N if and only if $x_i = g_{i,k} x_k$ on U for some $g_{i,k} \in \mathrm{SL}_2(C)$.

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- Let $T \subseteq C^n$ be a j-special variety and $U \subseteq C^n$ be a C-geodesic variety associated with T. Denote by $\langle\langle U, T \rangle\rangle$ the Zariski closure over C of the projection of the set

$$E^{\times}_{(z,J)}(K) \cap (U(K) \times T(K) \times K^2)$$

onto the last $3n$ coordinates.

- A D-*special* variety is a variety $S := \langle\langle U, T\rangle\rangle$ for some T and U as above. In this case S is said to be a D-special variety associated with T and U. We will also say that T (or U) is a j-special (respectively, geodesic) variety associated with S. A D-special variety associated with T is one associated with T and U for some C-geodesic variety U associated with T.
- $S \sim T$ means that S is a D-special variety associated with T. For a set Σ of j-special varieties $S \sim \Sigma$ means that $S \sim T$ for some $T \in \Sigma$.
- $\mathcal{S}_{\mathrm{D}}$ is the collection of all D-special varieties.
- D-special varieties are irreducible.
- Strongly J-special varieties are D-special.

Differential Modular Zilber–Pink with Derivatives

Definition

For a variety $V \subseteq C^{3n}$ we let the *D-atypical set* of V, denoted $\mathrm{Atyp}_{\mathrm{D}}(V)$, be the union of all D-atypical subvarieties of V, that is, atypical components of intersections $V \cap T$ where $T \subseteq C^{3n}$ is D-special.

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Theorem (A., 2019)

Let $(K;+,\cdot,D)$ be a differential field with an algebraically closed field of constants C. Given an algebraic variety $V \subseteq C^{3n}$, there is a finite collection Σ of proper j-special subvarieties of C^n such that

$$\mathrm{Atyp_D}(V)(K) \cap E_J^{\times}(K) \subseteq \bigcup_{\substack{P \sim \Sigma \\ P \in \mathcal{S}_{\mathrm{D}}}} P.$$

Pila and Scanlon proved some differential Zilber-Pink statements, but they did not consider derivatives.

Sketch of proof

- Use Seidenberg's embedding theorem. All solutions to the differential equation of j are of the form $j_g := j(gz)$ with $g \in \mathrm{GL}_2(\mathbb{C})$. Note that j_g is defined on $\mathbb{H}^g := g^{-1}\mathbb{H}$.

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- For a tuple $\bar{g} \in \mathrm{GL}_2(\mathbb{C})^n$ define functions $j_{\bar{g}}$ and $J_{\bar{g}}$, defined on $\mathbb{H}^{\bar{g}} := \mathbb{H}^{g_1} \times \cdots \times \mathbb{H}^{g_n}$.

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- Prove an analogue of Weak MZPD for $J_{\bar{g}}$-special varieties (uniform in $\bar{g}$).

Modular André-Oort with Derivatives

Conjecture (Pila)

For every algebraic variety $V \subsetneq \mathbb{C}^{3n}$ *there is a finite collection* Σ *of proper* $\mathbb{H}$*-special subvarieties of* $\mathbb{H}^n$ *such that every J-special subvariety of* V *is contained in a J-special variety of the form* $\langle\langle \bar{\gamma} U \rangle\rangle$ *for some* $\bar{\gamma} \in \mathrm{SL}_2(\mathbb{Z})^n$ *and some* $U \in \Sigma$*.*

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Theorem (A., 2018)

Let C *be an algebraically closed field of characteristic zero. Given an algebraic subvariety* $V \subsetneq C^{3n}$*, there is a finite collection* Σ *of proper j-special subvarieties of* C^n *such that every D-special subvariety of* V *is contained in a D-special variety associated with some* $T \in \Sigma$.

Note that Haden Spence also proved a weak version of MAOD which is different from the above theorem.

Thank you