A remark on unlikely intersections

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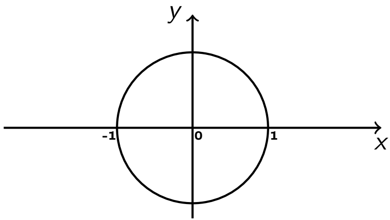

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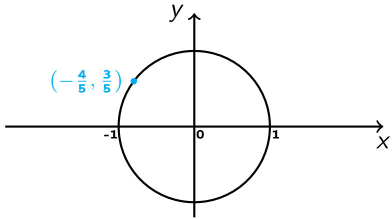

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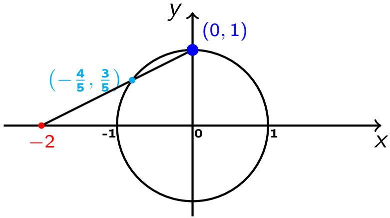

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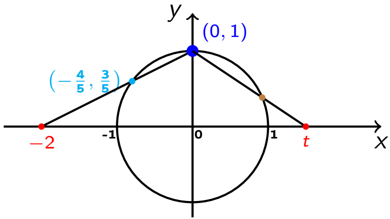

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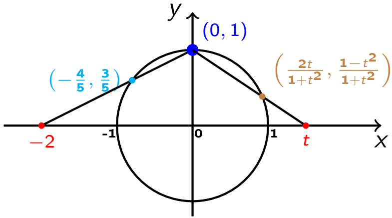

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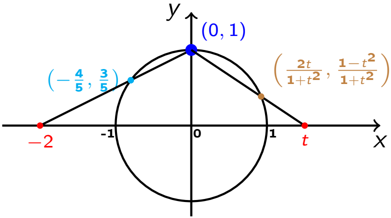

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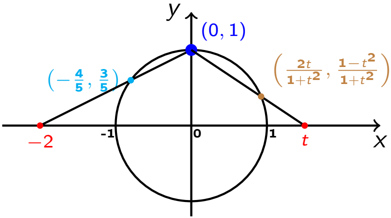

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- Examples of special points are roots of unity, i.e. numbers $z \in \mathbb{C}$ for which $z^n = 1$ for some $n > 0$ (e.g. $i^4 = 1$). These are the images of rational numbers under the function $e^{2\pi i z}$. Indeed, $\left(e^{2\pi i \cdot \frac{m}{n}}\right)^n = (e^{2\pi i})^m = 1$.

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- In other words, if a curve contains infinitely many points with special coordinates, then it must be of a special form.

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- The set $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ can be identified with the variety

$$\{(x, y) \in \mathbb{C}^2 : xy = 1\} \subseteq \mathbb{C}^2.$$

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- The variety defined by three equations $x^2 - y^2 = 1,\ x^2 - z^2 = 1,\ x(y - z) = 0$ has dimension 1 in $\mathbb{C}^3$.

Algebraic tori

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- Special varieties contain infinitely many special points. If an irreducible curve contains infinitely many special points, then it must be special.

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- If V is irreducible and contains a "Zariski dense" set of special points (too many special points) then V is special.

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- Two curves in a two-dimensional space are likely to intersect, while two curves in a three-dimensional space are not. If they do intersect, then we have an unlikely intersection.

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- Two curves in a two-dimensional space are likely to intersect, while two curves in a three-dimensional space are not. If they do intersect, then we have an unlikely intersection.

Theorem

Let $V, W \subseteq \mathbb{C}^n$ be irreducible algebraic varieties and $X \subseteq V \cap W$ be an irreducible component of the intersection. Then
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Definition

X is an atypical component of $V \cap W$ if $\dim X > \dim V + \dim W - n$.

Special and atypical subvarieties

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For example, if $V \subseteq (\mathbb{C}^\times)^2$ is defined by the equation $xy + x^2y^3 = i + 1$ then it contains the special variety defined by the equations $xy = i$, $x^2y^3 = 1$.

Conjecture on Intersections with Tori

Conjecture (CIT; Zilber, Bombieri-Masser-Zannier, Pink)

Every algebraic variety in $(\mathbb{C}^{\times})^n$ contains only finitely many maximal atypical subvarieties.

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Theorem (Weak CIT; Zilber, Bombieri-Masser-Zannier, Kirby)

Let V be an algebraic subvariety of $(\mathbb{C}^{\times})^n$. Then there is a finite collection Σ of proper algebraic subtori of $(\mathbb{C}^{\times})^n$ such that every atypical subvariety of V is contained in a (not necessarily torsion) coset of some $T \in \Sigma$.

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Theorem (A., 2019)

For every variety $V \subseteq (\mathbb{C}^{\times})^n$ there is a finite collection Σ of proper special subvarieties of $(\mathbb{C}^{\times})^n$ such that every atypical subvariety of V, whose weakly special closures is special, is contained in some $T \in \Sigma$.

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The proof combines weak CIT with Manin-Mumford.

Thank you

Γ-special and Γ-atypical sets

Definition

Let $\Gamma \subseteq (\mathbb{C}^{\times})^n$ be a subgroup of finite rank.

- A Γ-*special* subvariety of $(\mathbb{C}^{\times})^n$ is a translate of a torus by an element of Γ, i.e. a coset γT where T is a torus and $\gamma \in \Gamma$.
- For an algebraic variety $V \subseteq (\mathbb{C}^{\times})^n$, an atypical component X of an intersection $V \cap S$, where $S \subseteq (\mathbb{C}^{\times})^n$ is Γ-special, is called Γ-*atypical* if every coset of a subtorus of $(\mathbb{C}^{\times})^n$ containing X is Γ-special, i.e. contains a point of Γ. For example, if $X \cap \Gamma \neq \emptyset$ then X is Γ-atypical.

Mordell-Lang

Theorem (Mordell-Lang for $(\mathbb{C}^\times)^n$; Laurent)

Let $\Gamma \subseteq (\mathbb{C}^\times)^n$ be a subgroup of finite rank. Then an algebraic variety $V \subseteq (\mathbb{C}^\times)^n$ contains only finitely many maximal Γ-special subvarieties.

Theorem

If $V \cap \Gamma$ is Zariski dense in V then V is a finite union of Γ-special varieties.

Remark

The Mordell-Lang conjecture for abelian varieties, combined with the Mordell-Weil theorem, implies the Mordell conjecture, namely, a curve of genus ≥ 2 defined over $\mathbb{Q}$ has only finitely many rational points.

The Mordell-Lang conjecture for semi-abelian varieties was proven in a series of papers by Faltings, Vojta, Hindry, McQuillan, Raynaud, Laurent.

Weak CIT for Γ-special varieties

Theorem (A., 2019)

Let $\Gamma \subseteq (\mathbb{C}^\times)^n$ be a subgroup of finite rank. Then every subvariety $V \subseteq (\mathbb{C}^\times)^n$ contains only finitely many maximal Γ-atypical subvarieties.

Theorem (A., 2019)

Let $\Gamma \subseteq (\mathbb{C}^\times)^n$ be a subgroup of finite rank. Then for every subvariety $V \subseteq (\mathbb{C}^\times)^n$ there is a finite collection Σ of proper Γ-special subvarieties of $(\mathbb{C}^\times)^n$ such that every Γ-atypical subvariety of V is contained in some $T \in \Sigma$.

Sketch of proof

Lemma

Let $T \subseteq (\mathbb{C}^\times)^n$ be an algebraic torus and $V \subseteq (\mathbb{C}^\times)^n$ be an irreducible algebraic subvariety. Then the set

$$C := C_T := \{c \in (\mathbb{C}^\times)^n : V \cap cT \text{ is atypical in } (\mathbb{C}^\times)^n\}$$

is a proper Zariski closed subset of $(\mathbb{C}^\times)^n$.

Proof.

For every $c \in (\mathbb{C}^\times)^n$ obviously $\dim cT = \dim T$. Hence

$$C = \{c \in (\mathbb{C}^\times)^n : \dim(V \cap cT) \geq \dim V + \dim T - n + 1\}$$

which is Zariski closed in $(\mathbb{C}^\times)^n$.
One can show that a "generic" coset intersects V typically, hence $C \subsetneq (\mathbb{C}^\times)^n$. □

Sketch of proof (continued)

- We may assume V is irreducible.
- Let Σ_0 be the finite collection of algebraic subtori of $(\mathbb{C}^\times)^n$ given by Weak CIT.
- Pick a Γ-atypical subvariety X of V. Then $X \subseteq bT$ for some b and some $T \in \Sigma_0$.
- $bT \cap \Gamma \neq \emptyset$, hence $bT = \gamma T$ for some $\gamma \in \Gamma$.
- It can be shown that $V \cap \gamma T$ is atypical, hence $\gamma \in C_T$.
- Let Δ_T be the finite set of maximal Γ-special subvarieties of C_T. Observe that Δ_T consists of Γ-cosets of T.
- Then $\gamma \in A \in \Delta_T$ for some A.
- Therefore $X \subseteq \gamma T \subseteq AT = A$.
- Then $\Sigma = \bigcup_{T \in \Sigma_0} \Delta_T$ works.

Generalisations

- The same can be done for semi-abelian varieties.
- More generally, we can work inside a (Γ-)special variety S, define atypicality with respect to S and obtain analogous results in that setting. A component X of an intersection $V \cap T$, where $V, T \subseteq S$, is *atypical in S*, if

$$\dim X > \dim V + \dim T - \dim S.$$

- The modular j-function satisfies some functional equations that can be used to define special varieties, pose a modular analogue of CIT (which is a special case of the general Zilber-Pink conjecture), and prove similar weak statements there. There is a modular Mordell-Lang due to Habegger and Pila (2012), and an Ax-Schanuel for j due to Pila and Tsimerman (2015).
- All aforementioned results are also true uniformly in parametric families.