Strongly minimal sets in j-reducts of differentially closed fields

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- $j(gz) = j(z)$ for all $g \in \mathrm{SL}_2(\mathbb{Z})$.
- By means of j the quotient $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ is identified with $\mathbb{C}$ (thus, j is a bijection from the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ to $\mathbb{C}$).

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- Thus, given $z_1, z_2 \in \mathbb{H}$, we have $\Phi_N(j(z_1), j(z_2)) = 0$ if and only if $z_2 = gz_1$ for some $g \in \mathrm{GL}_2^+(\mathbb{Q})$.

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- The polynomials Φ_N are called *modular polynomials*.
- $\Phi_1(X, Y) = X - Y$ and all the other modular polynomials are symmetric.
- Two elements $w_1, w_2 \in \mathbb{C}$ are called *modularly independent* if they do not satisfy any modular relation $\Phi_N(w_1, w_2) = 0$.

Differential equation

- The j-function satisfies an order 3 algebraic differential equation over $\mathbb{Q}$. Namely, $\Psi(j, j', j'', j''') = 0$ where

$$\Psi(y_0, y_1, y_2, y_3) = \frac{y_3}{y_1} - \frac{3}{2}\left(\frac{y_2}{y_1}\right)^2 + \frac{y_0^2 - 1968y_0 + 2654208}{2y_0^2(y_0 - 1728)^2} \cdot y_1^2.$$

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- Thus

$$\Psi(y, y', y'', y''') = Sy + R(y)(y')^2,$$

where S denotes the *Schwarzian derivative* defined by $Sy = \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2$ and $R(y) = \frac{y^2 - 1968y + 2654208}{2y^2(y-1728)^2}$.

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- All functions $j(gz)$ with $g \in \mathrm{SL}_2(\mathbb{C})$ satisfy the differential equation $\Psi(y, y', y'', y''') = 0$ and in fact all solutions are of that form.

Two-variable equation

- In a differential field $(K; +, \cdot, ')$ for a non-constant $x \in K$ define a derivation $\partial_x : K \to K$ by $\partial_x : y \mapsto \frac{y'}{x'}$.

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- Let $\Psi(y, y', y'', y''') = 0$ be the differential equation of j. Consider its two-variable version

$$\chi(x, y) := \Psi(y, \partial_x y, \partial_x^2 y, \partial_x^3 y) = 0.$$

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$$\chi(x, y) := \Psi(y, \partial_x y, \partial_x^2 y, \partial_x^3 y) = 0.$$

- If we think of $x = x(t)$ as a complex function of t, then $y = j(gx(t))$ for some $g \in \mathrm{SL}_2(\mathbb{C})$.

Ax-Schanuel for j

Theorem (Ax-Schanuel for j; Pila-Tsimerman, 2015)

Let $(z_i, j_i) \in K^2,\ i = 1, \ldots, n$, be non-constant elements with with $\chi(z_i, j_i) = 0$. If j_i's are pairwise modularly independent then

$$td_C C(z_1, j_1, \partial_{z_1} j_1, \partial^2_{z_1} j_1, \ldots, z_n, j_n, \partial_{z_n} j_n, \partial^2_{z_n} j_n) \geq 3n + 1.$$

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Corollary (Ax-Schanuel without derivatives)

Let $(z_i, j_i) \in K^2$, $i = 1, \ldots, n$, be non-constant with $\chi(z_i, j_i) = 0$. If j_i's are pairwise modularly independent then $\operatorname{td}_C C(\bar{z}, \bar{j}) \geq n + 1$.

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Corollary (Ax-Lindemann-Weierstrass)

Let $z, j_i \in K,\ i = 1, \ldots, n$, be non-constant elements with $\chi(z, j_i) = 0$. If j_i's are pairwise modularly independent then $j_1, \partial_z j_1, \partial^2_z j_1, \ldots, j_n, \partial_z j_n, \partial^2_z j_n$ are algebraically independent over $C(z)$.

Strong minimality of $\Psi(y, y', y'', y''') = 0$

Theorem (Freitag-Scanlon, 2015)

Let $(K; +, \cdot, ')$ be a differentially closed field. Then the set $U \subseteq K$ defined by $\Psi(y, y', y'', y''') = 0$ is strongly minimal and geometrically trivial.

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Strongly minimal sets in differentially closed fields satisfy Zilber's trichotomy.

Theorem (Hrushovski-Sokolović, 1993)

A strongly minimal set in a differentially closed field must be either geometrically trivial (that is, the closure of a set is equal to the union of closures of singletons) or non-orthogonal to a Manin kernel (this is the locally modular non-trivial case) or non-orthogonal to the field of constants (this is the non-locally modular case).

Proof

- We need to show that every definable (possibly with parameters) subset V of U is either finite or co-finite.
- By stable embedding there is a finite subset $A = \{a_1, \ldots, a_n\} \subseteq U$ such that V is defined over A.
- It suffices to show that U realises a unique non-algebraic type over A.
- We know that $\mathrm{acl}(A) = (\mathbb{Q}\langle A\rangle)^{\mathrm{alg}} = (\mathbb{Q}(\bar{a}, \bar{a}', \bar{a}''))^{\mathrm{alg}}$.
- Let $u \in U \setminus \mathrm{acl}(A)$. Then u is modularly independent from each a_i.
- Assume WLOG that a_i's are pairwise modularly independent.

Proof continued

- By Ax-Lindemann-Weierstrass u, u', u'' are algebraically independent over $\mathbb{Q}\langle A \rangle$.
- Hence $\mathrm{tp}(u/A)$ is determined uniquely (axiomatised) by the set of formulae $\Psi(y, y', y'', y''') = 0$ and

$$\{P(y, y', y'') \neq 0 : P(Y_0, Y_1, Y_2) \in \mathbb{Q}\langle A \rangle [Y_0, Y_1, Y_2]\}.$$

- Similarly, if $A \subseteq U$ is a (finite) subset and $u \in U \cap \mathrm{acl}(A)$ then there is $a \in A$ such that $u \in \mathrm{acl}(a)$. This proves that U is geometrically trivial.
- Note that here we did not use full Ax-Schanuel, just the Ax-Lindemann-Weierstrass theorem.

ω-categoricity

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Order 1 strongly minimal sets in DCF_0 *are either non-orthogonal to* C *or are geometrically trivial and* ω*-categorical.*

The set U is not ω-categorical, since $j(gz) \in U$ and for $g \in \mathrm{GL}_2^+(\mathbb{Q})$ $j(gz)$ is algebraic over $j(z)$.

j-reducts of differential fields

If K is a differential field, consider its reduct $K_{E_j} := (K; +, \cdot, E_j)$ where $E_j(x, y)$ is a binary relation interpreted as the set of solutions of the equation $\chi(x, y) = 0$.

Basic axioms

The theory T_j^0 consists of the following first-order statements about a structure K in the language $\mathfrak{L}_j := \{+, \cdot, E_j, 0, 1\}$.

- A1 K is an algebraically closed field with an algebraically closed subfield $C := C_K$, which is defined by $E_j(1, y)$. Further, $C^2 \subseteq E_j$.
- A2 If $(z, j) \in E_j$ then for any $g \in \mathrm{SL}_2(C)$ we have $(gz, j) \in E_j$. Conversely, if for some j we have $(z_1, j), (z_2, j) \in E_j$ then $z_2 = gz_1$ for some $g \in \mathrm{SL}_2(C)$.
- A3 If $(z, j_1) \in E_j$ and $\Phi(j_1, j_2) = 0$ for some modular polynomial $\Phi(X, Y)$ then $(z, j_2) \in E_j$.
- AS Given a parametric family of varieties $(W_{\bar{c}})_{\bar{c} \in C} \subseteq K^{2n}$, there is a natural number $N(W)$ such that if $\bar{c} \in C$ satisfies $\dim W_{\bar{c}} \leq n$, and if $(\bar{z}, \bar{j}) \in E_j(K) \cap W_{\bar{c}}(K)$ and $j_i \notin C$ for all i, then $\Phi_N(j_i, j_k) = 0$ for some $N \leq N(W)$ and some $1 \leq i < k \leq n$.

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AS is an analogue of a uniform version of Ax-Schanuel. It holds in all differential fields (by the compactness theorem) and can be written as a first-order axiom scheme.

Predimension

- An E_j-*field* is a model K of T_j^0. By abuse of notation, for any n we let $E_j(K)$ denote the set of all tuples $(\bar{z}, \bar{j}) \in K^{2n}$ with $(z_i, j_i) \in E_j$ for all i.
- In an E_j-field one can define *predimension*. The predimension of a tuple $(\bar{z}, \bar{j})$, denoted $\delta(\bar{z}, \bar{j})$, is equal to $\operatorname{td}_{\mathbb{C}} \mathbb{C}(\bar{z}, \bar{j})$ minus the number of pairwise modularly independent j's. The latter is a dimension of trivial type.
- The AS axiom scheme states that the predimension is non-negative.
- An extension $A \subseteq B$ of E_j-fields is *strong* if for any tuple $(\bar{z}, \bar{j}) \in E_j(A)$ if we extend it to a tuple from B then the predimension does not go down.
- The class of E_j-fields has the strong amalgamation property, which allows one to carry out a Hrushovski style amalgamation-with-predimension construction and get a countable E_j-field U which is universal, saturated and homogeneous with respect to strong extensions.

Normal varieties

Let n be a positive integer, $k \leq n$ and $1 \leq i_1 < \ldots < i_k \leq n$. Denote $\bar{i} := (i_1, \ldots, i_k)$ and define the projection map $\mathrm{pr}_{\bar{i}} : K^{2n} \to K^{2k}$ by

$$\mathrm{pr}_{\bar{i}} : (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (x_{i_1}, \ldots, x_{i_k}, y_{i_1}, \ldots, y_{i_k}).$$

Definition

Let K be an algebraically closed field. An irreducible algebraic variety $V \subseteq K^{2n}$ is *normal* if for any $1 \leq i_1 < \ldots < i_k \leq n$ we have $\dim \mathrm{pr}_{\bar{i}} V \geq k$. We say V is *strongly normal* if the strict inequality $\dim \mathrm{pr}_{\bar{i}} V > k$ holds.

First-order theory of U

Consider the following statements for an E_j-field K.

EC For each normal variety $V \subseteq K^{2n}$ the intersection $E_j(K) \cap V(K)$ is non-empty.

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- EC For each normal variety $V \subseteq K^{2n}$ the intersection $E_j(K) \cap V(K)$ is non-empty.
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Let T_j be the theory A1-A3,AS,EC,NT.

Theorem (A., 2017)

T_j is the first-order theory of U. It is consistent and complete, ω-stable of Morley rank ω.

Existential Closedness

Let $(K; +, \cdot, ')$ be a countable saturated differentially closed field and $K_{E_j} = (K; +, \cdot, E_j)$ be its j-reduct.

Theorem (A., 2017)

The following are equivalent.

1. $U \cong K_{E_j}$.
2. $U \equiv K_{E_j}$.
3. K_{E_j} *satisfies EC.*

Conjecture (EC conjecture)

j-reducts of differentially closed fields satisfy EC. Hence, T_j is a complete axiomatisation of their first-order theory.

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Corollary

Assume the EC conjecture. Then all strongly minimal sets in K_{E_j} are either geometrically trivial or non-orthogonal to the field of constants.

Sketch of proof

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- U has Morley rank ω. Small sets, i.e. of finite Morley rank, are essentially existentially definable, while large sets are universally definable.
- Apply Ax-Schanuel as above.

Thank you