A remark on atypical intersections

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- The set $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ can be identified with the variety $\{(x, y) \in \mathbb{C}^2 : xy = 1\} \subseteq \mathbb{C}^2$.
- One can define the *Zariski topology* on $\mathbb{C}^n$ by declaring algebraic varieties to be the closed sets.

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- The variety defined by three equations $x^2 - y^2 = 1, \; x^2 - z^2 = 1, \; x(y - z) = 0$ has dimension 1.

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- A subvariety $T \subseteq \mathbb{G}_{\mathrm{m}}^{n}(\mathbb{C})$ (with coordinates $\bar{y}$) is called an *algebraic torus* if it is irreducible and is a subgroup of $\mathbb{G}_{\mathrm{m}}^{n}(\mathbb{C})$.

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- For any such subgroup the connected component of the identity element is an irreducible algebraic subgroup of finite index and is a torus. Every such group is equal to a disjoint union of a torus and its torsion cosets.

Special and atypical subvarieties

Definition

Irreducible components of algebraic subgroups of $\mathbb{G}_m^n(\mathbb{C})$, that is, torsion cosets of tori, are the *special* varieties. These are defined by equations of the form $y_1^{m_1} \cdots y_n^{m_n} = \zeta$ where ζ is a root of unity.

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Remark

If $T \subseteq V \subsetneq \mathbb{G}_{\mathrm{m}}^n$ and T is special then it is an atypical subvariety of V, for

$$\dim T > \dim V + \dim T - n.$$

Conjecture on Intersections with Tori

Conjecture (CIT; Zilber, Bombieri-Masser-Zannier, Pink)

Every algebraic variety in $\mathbb{G}_{\mathrm{m}}^n(\mathbb{C})$ contains only finitely many maximal atypical subvarieties.

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Every algebraic variety in $\mathbb{G}_{\mathrm{m}}^n(\mathbb{C})$ contains only finitely many maximal atypical subvarieties.

Conjecture (CIT)

Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n(\mathbb{C})$ be an algebraic variety. Then there is a finite collection Σ of proper special subvarieties of $\mathbb{G}_{\mathrm{m}}^n(\mathbb{C})$ such that every atypical subvariety X of V is contained in some $T \in \Sigma$.

Manin-Mumford conjecture

Theorem (Manin-Mumford for $\mathbb{G}_m^n$; Raynaud, Hindry)

A variety contains only finitely many maximal special subvarieties. In particular, an irreducible curve contains finitely many torsion points unless it is a torsion coset of a torus.

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Remark

Lang asked the following question in the 1960s. Assume $f(x, y) = 0$ contains infinitely many points (ξ, η) whose coordinates are roots of unity. What can be said about f?

Weak CIT

Theorem (Zilber, Kirby, Bombieri-Masser-Zannier)

Let V be an algebraic subvariety of $\mathbb{G}_{\mathrm{m}}^n$. Then there is a finite collection Σ of proper algebraic subtori of $\mathbb{G}_{\mathrm{m}}^n$ such that every atypical component of an intersection of V with an arbitrary coset of a torus is contained in a coset of some $T \in \Sigma$.

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Theorem (Ax, 1971)

If $f_1(\bar{z}), \ldots, f_n(\bar{z})$ are complex analytic functions defined on some open domain $U \subseteq \mathbb{C}^m$, and no $\mathbb{Q}$-linear combination of f_i's is constant, then

$$\mathrm{td}_{\mathbb{Q}}(f_1, \ldots, f_n, e^{f_1}, \ldots, e^{f_n}) \geq n + \mathrm{rk}\left(\frac{\partial f_i}{\partial z_j}\right).$$

Γ-special and Γ-atypical sets

Definition

Let $\Gamma \subseteq \mathbb{G}_{\mathrm{m}}^n$ be a subgroup of finite rank.

- A Γ-*special* subvariety of $\mathbb{G}_{\mathrm{m}}^n$ is a translate of a torus by an element of Γ, i.e. a coset γT where T is a torus and $\gamma \in \Gamma$.
- For an algebraic variety $V \subseteq \mathbb{G}_{\mathrm{m}}^n$, an atypical component X of an intersection $V \cap S$, where $S \subseteq \mathbb{G}_{\mathrm{m}}^n$ is Γ-special, is called Γ-*atypical* if every coset of a subtorus of $\mathbb{G}_{\mathrm{m}}^n$ containing X is Γ-special, i.e. contains a point of Γ. For example, if $X \cap \Gamma \neq \emptyset$ then X is Γ-atypical.

Mordell-Lang

Theorem (Mordell-Lang for $\mathbb{G}_{\mathrm{m}}^n$; Laurent)

Let $\Gamma \subseteq \mathbb{G}_{\mathrm{m}}^n$ be a subgroup of finite rank. Then an algebraic variety $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ contains only finitely many maximal Γ-special subvarieties.

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The Mordell-Lang conjecture for semi-abelian varieties was proven in a series of papers by Faltings, Vojta, Hindry, McQuillan, Raynaud, Laurent.

Weak CIT for Γ-special varieties

Theorem (A., 2019)

Let $\Gamma \subseteq \mathbb{G}_{\mathrm{m}}^n$ be a subgroup of finite rank. Then every subvariety $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ contains only finitely many maximal Γ-atypical subvarieties.

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Sketch of proof

Lemma

Let $T \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an algebraic torus and $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an irreducible algebraic subvariety. Then the set

$$C := C_T := \{c \in \mathbb{G}_{\mathrm{m}}^n : V \cap cT \text{ is atypical in } \mathbb{G}_{\mathrm{m}}^n\}$$

is a proper Zariski closed subset of $\mathbb{G}_{\mathrm{m}}^n$.

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Proof.

For every $c \in \mathbb{G}_m^n$ obviously $\dim cT = \dim T$. Hence

$$C = \{c \in \mathbb{G}_m^n : \dim(V \cap cT) \geq \dim V + \dim T - n + 1\}$$

which is Zariski closed in $\mathbb{G}_m^n$.
One can show that a "generic" coset intersects V typically, hence $C \subsetneq \mathbb{G}_m^n$. □

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- Pick a Γ-atypical subvariety X of V. Then $X \subseteq bT$ for some b and some $T \in \Sigma_0$.
- $bT \cap \Gamma \neq \emptyset$, hence $bT = \gamma T$ for some $\gamma \in \Gamma$.

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- Then $\gamma \in A \in \Delta_T$ for some A.
- Therefore $X \subseteq \gamma T \subseteq AT = A$.
- Then $\Sigma = \bigcup_{T \in \Sigma_0} \Delta_T$ works.

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- More generally, we can work inside a (Γ-)special variety S, define atypicality with respect to S and obtain analogous results in that setting. A component X of an intersection $V \cap T$, where $V, T \subseteq S$, is *atypical in* S, if

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- All aforementioned results are also true uniformly in parametric families.

Thank you